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# NOTES ON MONADIC LOGIC. PART A. MONADIC THEORY OF THE REAL LINE†

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ABSTRACT
The second-order theory of the continuum in the Cohen extension of a set-theoretic universe is interpreted in the monadic theory of the real line and may be interpreted in the monadic topology of Cantor's discontinuum as well.

## Introduction

Assuming the Continuum Hypothesis (CH), Shelah [Sh 42] proved the undecidability of the monadic second-order theory of the real line by interpreting true first-order arithmetic in it. But the monadic theory of the real line happens to be more expressive ([Gu 2], [GuSh 123], [GuSh 143]). In the last of the three papers, the second-order theory of the continuum in the Cohen extension of the universe has been interpreted, under CH, in the monadic theory of the real line as well as the monadic theory of any non-modest short chain. In this paper, we get rid of CH.

To simplify the exposition, we treat the case of the real line only. For the reader's convenience, the proof is self-contained. It is based on the notes of lectures in Rutgers and Jerusalem in Fall 1986.

NOTATION. (We work in the topological space ${}^{\omega}\omega$ rather than in the standard real line.)

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$^{\omega}\omega$ has the Tychnonov topology (considering $\omega$ as a discrete space).

$u, v$ denote non-empty regular open subsets (of $^{\omega}\omega$, i.e., they are equal to the interior of their closure).

cl($A$) is the closure of $A$.

$M_X$ (for a topological space $X$) is the model $(\mathscr{P}(X), \subseteq, \mathrm{cl})$ where $\mathscr{P}(X)$ is the power set of $X$, $\subseteq$ is inclusion, and cl is the closure operation.

We let $X$, $Y$, $Z$, $T$ be monadic variables for a subset of $^{\omega}\omega$, $X \equiv Y$ iff their symmetric difference is nowhere dense and $X \subseteq^* Y$ if $X - Y$ is nowhere dense.

## §1. The basic interpretation

1.1. Definition. (1) For any formula $\varphi(u, \bar{a})$ (not necessarily monadic) let

$$\mathrm{val}_u \varphi(u, \bar{a}) = \bigcup \{ u : u \text{ open regular subsets of } {}^{\omega}\omega, \text{ and } {}^{\omega}\omega \models \varphi(u, \bar{a})\}.$$

(2) We call $\varphi(u, \bar{a})$ *regular* if $\mathrm{val}_u \varphi(u, \bar{a})$ is open regular; $\varphi(u, \bar{z})$ is regular if every $\varphi(u, \bar{a})$ is.

(3) We call $\varphi(u, X_1, \ldots, X_k; \bar{a})$ *regular* (in $u, X_1 \ldots, X_k$) if

$$\forall u \, \forall X_1, \ldots, X_k \, \forall X'_1, \ldots, X'_k \left[ \bigwedge_{l=1}^{k} X_l \cap u \equiv X'_l \cap u \rightarrow \varphi(u, X_1, \ldots, X_k; \bar{a}) \equiv \varphi(u, X'_1, \ldots, X'_k; \bar{a}) \right].$$

1.1A. Observation. (1) $\varphi'(u, \bar{a}) = (\forall u' \subseteq u)(\exists u'' \subseteq u')\varphi(u'', \bar{a})$ is always regular and $\mathrm{val}_u \varphi(u, \bar{a}) \equiv \mathrm{val}_u \varphi'(u, \bar{a})$ for every $\bar{a}$.

(2) $\varphi'(u, X_1, \ldots, X_k; \bar{y}) \stackrel{\mathrm{def}}{=} (\forall u' \subseteq u) \; (\exists u'' \subseteq u') \; (\exists X'_1, \ldots, X'_k) [\bigwedge_{l=1}^{k} X'_l \cap u'' \equiv X_l \cap u'' \wedge \varphi(u'', X'_1, \ldots, X'_k; \bar{y})]$ is always regular in $u, X_1 \ldots, X_k$.

(3) We will assume without saying that we regularize our formulas this way.

1.2. Lemma. *There are regular monadic formulas $\psi_a(u, \ldots)$, $\psi_b(u, \ldots)$, $\psi_c(u, \ldots)$ and a sequence $\langle D^r_i : i < 2^{\aleph_0} \rangle$ of dense countable pairwise disjoint subsets of $^{\omega}\omega$ (we let $D^r = \bigcup \{D^r_i : i < 2^{\aleph_0}\}$) such that:*

(1) *for some $W_a \subseteq {}^{\omega}\omega - D^r$, for every $X \subseteq D^r$,*

$$\mathrm{val}_u \psi_a(u, X; D^r, W_a) \equiv \mathrm{val}_u \left( \bigvee_i u \cap X \subseteq u \cap D^r_i \right),$$

(2) *for some $W_a \subseteq {}^{\omega}\omega - D^r$, for every $X \subseteq D^r$,*

$$\mathrm{val}_{u}\,\psi_b(u, X; D^r, W_a) \equiv \mathrm{val}_{u}\Big(\bigvee_i \ u \cap X = u \cap D_i^r\Big),$$

(3) *for every symmetric two-place function* $R$ *from* $\{i : i < 2^{\aleph_0}\}$ *to* $\{u \subseteq {}^{\omega}\omega : u$ *open regular*$\} \cup \{\emptyset\}$, for some *subset* $W_R$ *of* ${}^{\omega}\omega$ (*and* $W_a$ *as in* (2)), *for every* $X, Y \subseteq D^r$,

$$\mathrm{val}_{u}\,\psi_c(u, X, Y; D^r, W_a, W_R) \equiv \bigcup \{u\text{: for some } i \neq j,\ u \subseteq R(i,j),\ u \cap X = u \cap D_i^r,\ u \cap Y = u \cap D_j^r\}.$$

PROOF. This is presented in Section 3.
Note we can agree $R(i, i) = \emptyset$.

1.3. CONVENTION. Let $\langle D_i^r : i < 2^{\aleph_0}\rangle$ be as in 1.2, $W_a$, $W_R$ be as in 1.2(2), 1.2(3) respectively. For $R$ a symmetric two-place relation on $\{i : i < 2^{\aleph_0}\}$, we identify it with the function $R'$:

$$R'(i,j) = \begin{cases} {}^{\omega}\omega & \text{if } i \neq j,\ \vDash R[i,j] \\ \emptyset & \text{otherwise.} \end{cases}$$

1.4. CLAIM. There is a finite sequence $\bar{W}^0$ (of subsets of ${}^{\omega}\omega$) and regular formulas $\varphi_{\mathrm{nu}}, \varphi_{\mathrm{ze}}, \varphi_{\mathrm{suc}}, \varphi_{\mathrm{ad}}, \varphi_{\mathrm{ord}}, \varphi_{\mathrm{ml}}$ such that:

(1) for $X \subseteq {}^{\omega}\omega$, $\mathrm{val}_{u}[\varphi_{\mathrm{nu}}(u, X; \bar{W}^0)] \equiv \bigcup\{u\text{: for some } k,\ u \cap X = u \cap D_k^r\}$

[the intended meaning is that $X$ represents a natural number].

(2) for $X \subseteq {}^{\omega}\omega$, $\mathrm{val}_{u}[\varphi_{\mathrm{ze}}(u, X; \bar{W}^0)] \equiv \bigcup\{u : u \cap X = u \cap D_0^r\}$

[the intended meaning is that $X$ represents zero].

(3) $$\mathrm{val}_{u}[\varphi_{\mathrm{suc}}(u, X, Y; \bar{W}^0)] \equiv \bigcup\{u : \text{for some } k,\ u \cap X = u \cap D_k^r,\ u \cap Y = u \cap D_{k+1}^r\}$$

[the intended meaning is that $Y$ is the successor of $X$, i.e., the corresponding numbers are like that].

(4) $$\mathrm{val}_{u}[\varphi_{\mathrm{ad}}(u, X_1, X_2, X_3; \bar{W}^0)] \equiv \bigcup \Big\{u : \text{for some } k_1, k_2, k_3 < \omega,\ k_3 = k_2 + k_1 \text{ and } \bigwedge_{l=1}^{3} u \cap X_l = u \cap D_{k_l}^r\Big\}$$

[the intended meaning is addition].

(5) $$\mathrm{val}_u[\varphi_{\mathrm{ord}}(u, X; \bar{W}^0)] = \bigcup\{u : \text{for some } i,\ u \cap X = u \cap D_i^r\}$$

[the intended meaning is that $X$ is an ordinal, i.e., represents one].

(6) $$\mathrm{val}_u[\varphi_{\mathrm{ml}}(u, X_1, X_2, X_3; \bar{W}^0)] \equiv \bigcup\left\{u : \text{for some } k_1, k_2, k_3 < \omega,\ k_3 = k_2 \times k_1 \text{ and } \bigwedge_{l=1}^{3} u \cap X_l = u \cap D_{k_l}^r\right\}$$

[the intended meaning is multiplication].

PROOF. Easy (was done in [Sh 42]), but here are some new details. Let

$$R_{\mathrm{suc}}^1 = \{\{k, \omega + k\} : k < \omega\},$$

$$R_{\mathrm{suc}}^2 = \{\{k+1, \omega + k\} : k < \omega\},$$

and, for $l = 1, 2, 3$,

$$R_{\mathrm{ad}}^l = \{\{\omega^3(1+k_3) + \omega^2(1+k_2) + \omega(1+k_1), k_l\} : k_3 = k_2 + k_1 \text{ are natural numbers}\},$$

$$R_{\mathrm{ml}}^l = \{\{\omega^3(1+k_3) + \omega^2(1+k_2) + \omega(1+k_1), k_l\} : k_3 = k_2 \times k_1 \text{ are natural numbers}\}.$$

Let $W_{\mathrm{suc}}^m = W_{R_{\mathrm{suc}}^m}$, $W_{\mathrm{ad}}^l = W_{R_{\mathrm{ad}}^l}$, $W_{\mathrm{ml}}^l = W_{R_{\mathrm{ml}}^l}$ for $m = 1, 2$, $l = 1, 2, 3$. Let

$$D_N^r = \bigcup_{n<\omega} D_n^r,$$

$$\bar{W}^0 = \langle D^r, D_0^r, D_N^r, W_a, W_{\mathrm{suc}}^1, W_{\mathrm{suc}}^2, W_{\mathrm{ad}}^l, W_{\mathrm{ml}}^l\rangle_{l=1,2,3}.$$

Now we let

$$\varphi_{\mathrm{nu}}(u, X; \bar{W}^0) \stackrel{\mathrm{def}}{=} [u \cap X \subseteq D_N^r \wedge \psi_b(u, u \cap X; W_a)],$$

$$\varphi_{\mathrm{ze}}(u, X; \bar{W}^0) \stackrel{\mathrm{def}}{=} [u \cap X = u \cap D_0^r],$$

$$\varphi_{\mathrm{suc}}(u, X, Y; \bar{W}^0) = [\varphi_{\mathrm{nu}}(u, X; \bar{W}^0) \wedge \varphi_{\mathrm{nu}}(u, Y; \bar{W}^0) \wedge (\exists Z)[\varphi_{\mathrm{ord}}(u, Z; \bar{W}^0) \wedge u \cap Z \subseteq D^r - D_N^r \wedge \psi_c(u, X, Z; W_{R_{\mathrm{suc}}^1}) \wedge \varphi_c(u, Y, Z; W_{R_{\mathrm{suc}}^2})]].$$

Similarly for $\varphi_{\mathrm{ad}}$ and $\varphi_{\mathrm{ml}}$.

1.5. DEFINITION. Let the monadic formula $\theta_1(u, \bar{T})$ say that $\bar{T}$ satisfies all reasonable properties of what $\bar{W}^0$ satisfies in $u$ (we delay the question of "every natural number is standard"), i.e., $\lg(\bar{T}) = \lg(\bar{W}^0)$ and $\theta_1$ is the conjunction of the following formulas (all saying what occurs inside $u$ only):

(1) every natural number has a unique successor, i.e.,
$(\forall X)[\varphi_{nu}(u, X, \bar{T}) \to (\exists Y)\varphi_{suc}(u, X, Y, \bar{T})]$ and
$(\forall X)(\forall Y_1)\forall Y_2[\varphi_{suc}(u, X, Y_1) \wedge \varphi_{suc}(u, X, Y_2, T) \to Y_1 \cap u \equiv Y_2 \cap u]$;

(2) a natural number is a successor iff it is not zero;

(3) every pair of natural numbers has a unique sum;

(4) every pair of natural numbers has a unique product;

(5) $x + (y + 1) = (x + y) + 1$, $x + 0 = x$;

(6) $x \times (y + 1) = x \times y + x$, $x \times 0 = 0$;

(7) addition and product are commutative;

(8) $x + 1 = y + 1$ implies $x = y$.

1.5A. Convention. Omitting $u$ in $\theta_1$ means taking ${}^{\omega}\omega$. Similarly everywhere else.

1.6. Claim. (1) $\models \theta_1[\bar{W}^0]$ (for the $\bar{W}^0$ from Claim 1.4).

(2) If $\models \theta_1[u, \bar{W}]$ then we can find $D_n$ $(n < \omega)$ pairwise disjoint such that:

(a) $\varphi_{ze}(u, D_0; \bar{W})$,

(b) $\varphi_{nu}(u, D_n; \bar{W})$,

(c) $\varphi_{suc}(u, D_n, D_{n+1}; \bar{W})$,

(d) $\varphi_{ad}(u, D_n, D_m, D_{m+n}; \bar{W})$,

(e) $\varphi_{ml}(u, D_n, D_m, D_{m\times n}; \bar{W})$.

(3) If $D'_n$ $(n < \omega)$ satisfies (a), (b), (c) then $\bigwedge_{n<\omega} D_n \cap u \equiv D'_n \cap u$.

Proof. Easy.

As we have said, we desire to express "$\bar{W}^0$ code standard natural number only".

1.7. Definition. Let $\theta_2(u, \bar{Y})$ say that, hereditarily in $u$:

$$\theta_1(u, \bar{Y}) \wedge \neg(\exists \bar{Y}', u')[u' \subseteq u \wedge \theta^a(u', \bar{Y}', \bar{Y}) \wedge \theta^b(u', \bar{Y}', \bar{Y})]$$

where

$$\theta^b(u, \bar{Y}', \bar{Y}) \stackrel{\text{def}}{=} \theta_1(u, \bar{Y}') \wedge (\forall u, X_1, X_2)[\varphi_{suc}(u, X_2, X_1; \bar{Y}') \to (\exists X'_1 \supseteq X_1)(\exists X'_2 \supseteq X_2)\varphi_{suc}(u, X'_1, X'_2; \bar{Y})],$$

$$\theta^a(u, \bar{Y}', \bar{Y}) \stackrel{\text{def}}{=} (\forall X)[\varphi_{nu}(u, X; \bar{Y}') \to (\exists X')(X \subseteq X')\varphi_{nu}(u, X'; \bar{Y})].$$

1.8. Claim. (1) $\models \theta_2[\bar{W}^0]$.

(2) If $\models \theta_2[\bar{W}]$ and $D_n$ $(n < \omega)$ are as in 1.6 *then* for every $X$

$$\mathrm{val}_{\mathcal{U}}\varphi_{\mathrm{nu}}(\mathcal{U}, X; \bar{W}) \equiv \mathrm{val}_{\mathcal{U}} \bigvee_n (\mathcal{U} \cap X = \mathcal{U} \cap D_n).$$

(3) The parallel of (2) holds for $\theta_2(\mathcal{U}, \bar{W})$.

PROOF. (1) Immediate.

(2) If not, let $X \subseteq {}^\omega\omega$ be such that some open regular $\mathcal{U}_0$ is disjoint from $\mathrm{val}_{\mathcal{U}}(\mathcal{U} \cap X = \mathcal{U} \cap D_n)$ for every $n$ but $\mathcal{U}_0 \subseteq \mathrm{val}_{\mathcal{U}}\varphi_{\mathrm{nu}}(\mathcal{U}, X; \bar{W})$.

Fix $\mathcal{U}_0$, $X$. We define by induction on $n$, $X_n \subseteq \mathcal{U}_0$ such that $\mathcal{U}_0 \subseteq \mathrm{val}_{\mathcal{U}}\, \varphi(\mathcal{U}, X_n; \bar{W})$, $\mathcal{U}_0 \subseteq \mathrm{val}_{\mathcal{U}}\, \varphi_{\mathrm{suc}}(\mathcal{U}, X_n, X_{n+1}; \bar{W})$ and $\mathcal{U}_0$ is disjoint from $\mathrm{val}_{\mathcal{U}}(\mathcal{U} \cap X_n = \mathcal{U} \cap D_m)$ for every $m$ ($X_0 = X$, of course). Let $X'_n \subseteq X_n$ be countable and dense in $\mathcal{U}_0$, $\bigwedge_{k<n} X'_n \cap X'_k = \varnothing$. There is an autohomeomorphism $F$ of ${}^\omega\omega$ taking $D'_n$ to $X'_n$ for $n < \omega$ and $\mathcal{U}_0$ to itself (Cantor Theorem). Now $F(\bar{W}^0)$, $\mathcal{U}_0$ can serve as $\bar{Y}'$, $\mathcal{U}$ contradicting the second part of $\theta_2(\bar{W})$.

1.8A. REMARK. Applying this to other topological spaces, we can replace Cantor Theorem by strengthening of 1.2. Similarly in 2.10.

## §2. Interpreting the universe after forcing

2.1. DEFINITION. Let $Q$ be the forcing notion: open regular subsets of ${}^\omega\omega$, *with the order: the converse of inclusion (this is the Cohen forcing).*

2.1A. CONVENTION. $\bar{W}$ denotes a sequence such that $\models \theta_2[\bar{W}]$, $D_n(\bar{W})$ ($n < \omega$) are as in 1.2, $D(W) = \langle D_n(W) : n < \omega \rangle$. In this section $D$ denotes a $\omega$-sequence of dense pairwise disjoint subsets of ${}^\omega\omega$.

2.2. DEFINITION. (1) We say that $X$ $\bar{D}$-represents in $\mathcal{U}_0$ a $Q$-name $\underset{\sim}{n}$ of a natural number if:

(a) $\mathcal{U}_0 \equiv \mathcal{U}_0 \cap \mathrm{val}_{\mathcal{U}}\varphi_{\mathrm{nu}}(\mathcal{U}, X; \bar{W})$,

(b) $\Vdash_Q$ "$\underset{\sim}{n}$ is a natural number",

(c) for every $k < \omega$ and $\mathcal{U} \subseteq \mathcal{U}_0$

$$\mathcal{U} \Vdash_Q \text{“}\underset{\sim}{n} = k\text{”} \qquad \text{iff } \mathcal{U} \cap X \equiv \mathcal{U} \cap D_k.$$

(2) If $\mathcal{U}_0 \equiv {}^\omega\omega$ we omit it.

2.3. CLAIM. (1) Suppose $\models \theta_2[\bar{W}]$. $\models \varphi_{\mathrm{nu}}(\mathcal{U}, X; \bar{W})$ *iff* $X$ $\bar{D}(\bar{W})$-represents in $\mathcal{U}$ some $Q$-name of a natural number.

(2) If $X$ $\bar{D}$-represents in $\mathcal{U}$ a $Q$-name $\underset{\sim}{n}$ (of a natural number), then $X$ $D$-represents $\underset{\sim}{n}$ in every $\mathcal{U}' \subseteq \mathcal{U}$.

PROOF. Suppose $\models \varphi_{\mathrm{nu}}(\mathcal{U}, X; \bar{W})$. We know that

$$K = \{ v : v \subseteq u \text{ and } v \cap X = v \cap \bar{D}_n(\bar{W}) \text{ for some } n = n(v)\}$$

is such that

(*) $(\forall u' \subseteq u)(\exists u'' \subseteq u')(u'' \in K)$.

Let $\{ v_\alpha : \alpha < A^0\}$ be a maximal subset of $K$ such that any two members are disjoint. Clearly $\bigcup_\alpha v_\alpha$ is a dense subset of $u$ [by (*)]. We define $\underset{\sim}{n}$ by

$$\underset{\sim}{n} \textit{ is } n \text{ if, for some } \alpha,\ v_\alpha \text{ is in } \underset{\sim}{G}_Q \text{ and } v_\alpha \cap \bar{D}_n(\bar{W}) = v_\alpha \cap X$$

$$(\underset{\sim}{G}_Q \text{ is the generic set) and zero otherwise.}$$

Easily, $\underset{\sim}{n}$ is a $Q$-name of a natural number and $X$ $\bar{D}(\bar{W})$-represents it in $u$. The other direction is easy too.

2.4. Claim. Suppose $\models \theta_2[u, \bar{W}]$. If $\underset{\sim}{n}$ is a $Q$-name and $u \Vdash_Q$ "$\underset{\sim}{n}$ a natural number", *then* some $X$ $\bar{D}(\bar{W})$-represents $\underset{\sim}{n}$ in $u$.

Proof. Let $\{ u_\alpha : \alpha < \alpha_0\}$ be a maximal antichain of members of $Q$, $u_\alpha \subseteq u$, $u_\alpha$ force a value to $\underset{\sim}{n}$, say $n(\alpha)$. So $\{ u_\alpha : \alpha < \alpha_0\}$ is a family of pairwise disjoint regular open subsets of $u$. Let $X = \bigcup_\alpha (u_\alpha \cap \bar{D}_{n(\alpha)}(\bar{W}))$.

2.5. Claim. Suppose $\models \theta_2[\bar{W}]$. If for $l = 1, 2, 3$, $X_l$ $\bar{D}(\bar{W})$-represents in $u$ the $Q$-name $\underset{\sim}{n}_l$ of a natural number, then for every $v \subseteq u$:

(a) $v \Vdash_Q$ "$\underset{\sim}{n}_1 = 0$" *iff* $\varphi_{\text{ze}}(v, X_1; \bar{W})$,

(b) $v \Vdash_Q$ "$\underset{\sim}{n}_1 = \underset{\sim}{n}_2$" *iff* $v \cap X_1 \equiv v \cap X_2$,

(c) $v \Vdash_Q$ "$\underset{\sim}{n}_1 + 1 = \underset{\sim}{n}_2$" *iff* $\varphi_{\text{suc}}(v, X_1, X_2; \bar{W})$,

(d) $v \Vdash_Q$ "$\underset{\sim}{n}_1 + \underset{\sim}{n}_2 = \underset{\sim}{n}_3$" *iff* $\varphi_{\text{ad}}(v, X_1, X_2, X_3; \bar{W})$,

(e) $v \Vdash_Q$ "$\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = \underset{\sim}{n}_3$" *iff* $\varphi_{\text{ml}}(v, X_1, X_2, X_3; \bar{W})$.

Proof. Easy (from definition).

Next we deal with reals, i.e., sets of natural numbers.

2.6. Definition. We say that $Y$ $\bar{D}$-represents in $u$ a $Q$-name $\underset{\sim}{a}$ of a set of natural numbers if, for every $v \subseteq u$ and $k < \omega$,

(a) $v \Vdash_Q$ "$k \in \underset{\sim}{a}$" *iff* $v \cap D_k \subseteq^* v \cap Y$,

(b) $v \Vdash_Q$ "$k \notin \underset{\sim}{a}$" *iff* $v \cap Y \cap D_k \equiv \varnothing$.

2.7. Definition. $\varphi_{\text{rl}}(u, Y; \bar{W})$ is

$$(\forall v \subseteq u)(\forall X)[\varphi_{\text{nu}}(v, X; \bar{W}) \rightarrow (\exists v' \subseteq v)(v' \cap X \subseteq Y \vee v' \cap X \cap Y = \varnothing)].$$

2.8. Claim. Suppose $\models \theta_2[u, \bar{W}]$.

(1) $\models \varphi_{\mathrm{rl}}(u, Y; \bar{W})$ *iff* $Y$ $\bar{D}(\bar{W})$-represents in $u$ some $Q$-name of a set of natural numbers.

(2) Every $Q$-name $\underset{\sim}{a}$ of a set of natural numbers is $\bar{D}(\bar{W})$-represented by some $Y$.

(3) If $Y$ $\bar{D}$-represents in $u$ a $Q$-name $\underset{\sim}{a}$ of a set of reals *then* $Y$ $\bar{D}$-represents $\underset{\sim}{a}$ in every $u' \subseteq u$.

PROOF. (1) Suppose $\models \varphi_{\mathrm{rl}}(u, \bar{Y}, \bar{W})$. Define $\underset{\sim}{a}$ by:

(*) $k \in \underset{\sim}{a}$ *iff* there is $v \subseteq u$, $v \in \underset{\sim}{G}_Q$ (the generic subset) and $v \cap D_k(\bar{W}) \subseteq Y$.

It is easy to check that it is as required. For the other direction suppose $X$ $\bar{D}(\bar{W})$-represents in $u$ some $Q$-name $\underset{\sim}{a}$ of a set of natural numbers. Now for every $v \subseteq u$ and $X$ such that $\varphi_{\mathrm{nu}}(v, X; \bar{W})$, first find $u_0 \subseteq v$ and $k$ such that $u_0 \cap X = u_0 \cap D_k(\bar{W})$ (see choice of $\varphi_{\mathrm{nu}}$), next choose $v_1 \subseteq u_0$ such that $v_1 \Vdash_Q$ "$k \in \underset{\sim}{a}$" or $v_1 \Vdash_Q$ "$k \notin \underset{\sim}{a}$". If the former holds then (by Definition 2.6) $v_1 \cap D_k(\bar{W}) \subseteq^* v_1 \cap X$ hence, for some $v' \subseteq v_1$, $v' \cap D_k(\bar{W}) \subseteq v_1 \cap X$; so $v'$ is as required in the definition of $\varphi_{\mathrm{rl}}$. If the latter ($v_1 \Vdash_Q$ "$k \notin \underset{\sim}{a}$") holds, then (by Definition 2.6) $v_1 \cap D_k(\bar{W}) \cap X \equiv \varnothing$; so for some $v' \subseteq v$, $v' \cap D_k(\bar{W}) \cap X = \varnothing$ and so $v'$ is as required in the definition of $\varphi_{\mathrm{rl}}$.

(2) Let, for each $k$, $\langle u^k_\alpha : \alpha < \alpha_k \rangle$ be a maximal antichain of $Q$, such that $u^k_\alpha \Vdash_Q$ "$k \in \underset{\sim}{a}$" or $u^k_\alpha \Vdash_Q$ "$k \notin \underset{\sim}{a}$". Let

$$Y = \bigcup \{ D_k(\bar{W}) \cap u^k_\alpha : k < \omega, \alpha < \alpha_k, u^k_\alpha \Vdash_Q \text{"} k \in \underset{\sim}{a} \text{"} \}.$$

As $\langle D_n(\bar{W}) : n < \omega \rangle$ are pairwise disjoint, $Y$ is as required.

(3) Trivial.

2.9. CLAIM. Assume $\models \theta_2[\bar{W}]$, $X$ $\bar{D}(\bar{W})$-represents in $u$ the $Q$-name $\underset{\sim}{n}$ of a natural number, and $Y$ $\bar{D}(\bar{W})$-represents in $u$ the $Q$-name $\underset{\sim}{a}$ of a real. Then for $v \subseteq u$:

$$v \cap X \subseteq^* Y \qquad \textit{iff } v \Vdash \text{"} \underset{\sim}{n} \in \underset{\sim}{a} \text{"}.$$

PROOF. Check definitions.

2.10. DEFINITION. We say that $\bar{W}^+ = \bar{W}^{*\frown}\langle W \rangle$ $\bar{D}$-represents in $u$ a $Q$-name $\underset{\sim}{A}$ of a set of reals if:

(a) $u \Vdash_Q$ "$\underset{\sim}{A}$ is a set of reals";

(b) $\models \theta_2[u, \bar{W}^*]$;

(c) $u \cap D_n(\bar{W}^*) \subseteq^* D_n$;

(d) TFAE for all $Q$-names $\underset{\sim}{a}$ of a real and $v \subseteq u$:

 (α) $v \Vdash_Q$ "$\underset{\sim}{a} \in \underset{\sim}{A}$",

 (β) there is $X$ such that

(i) $\vDash \varphi_{\rm ord}(v, X; \bar{W}^*) \wedge (\forall v' \subseteq v) \neg \varphi_{\rm nu}(v', X; \bar{W}^*)$,

(ii) for every $v' \subseteq v$, and $k < \omega$,

$$v' \Vdash_Q \text{"} k \in \underset{\sim}{a} \text{"} \qquad \textit{iff} \vDash \psi_c(v', D_k(\bar{W}^*), X; D^*, W_a^*, W)$$

$(D^*, W_a^*$ — from $\bar{W}^*)$.

REMARK. On $\psi_c$ see 1.2(3).

2.10A. CLAIM. Suppose $\vDash \theta_2[u, \bar{W}^*]$. For every $Q$-name $\underset{\sim}{A}$ of a set of reals there is $W$ such that $\bar{W}^+ = \bar{W}^{*\wedge}\langle W \rangle$ $\bar{D}(\bar{W}^*)$-represents $\underset{\sim}{A}$ in $u$.

PROOF. Choose countable dense $D_n' \subseteq D_n$, and so there is an autohomeomorphism $F$ of ${}^\omega\omega$ taking $D_n''$ to $D_n'$. Let $\{\underset{\sim}{a}_\alpha : \alpha < 2^{\aleph_0}\}$ list all $Q$-names of reals. For each $\alpha$, $k$ let $\{u_{\alpha,\zeta}^k : \zeta < \zeta_\alpha\}$ be a maximal set of pairwise disjoint members of $Q$ such that

$$u_{\alpha,\zeta}^k \Vdash_Q \text{"}\underset{\sim}{a}_\alpha \in \underset{\sim}{A} \text{ and } k \in \underset{\sim}{a}_\alpha\text{"}.$$

Define a two-place function $R$ from $2^{\aleph_0}$ to $Q$:

$$R(\omega+\alpha, k) = R(k, \omega+\alpha) \equiv \bigcup\{u_{\alpha,\zeta}^k : \zeta < \zeta_\alpha\}$$

(i.e., the interior of the closure of this union) and

$$R(\alpha, \beta) = \emptyset \qquad \text{when } (\alpha < \omega \wedge \beta < \omega) \vee (\alpha \geqq \omega \wedge \beta \geqq \omega).$$

Now we apply 1.2(3) and get $W_R$. Lastly $W \stackrel{\rm def}{=} W_R$ is as required.

2.11. DEFINITION. Let $\varphi_{\rm srl}(u, \bar{W}^+, \bar{W})$ (where $\bar{W}^+ = \bar{W}^{*\wedge}\langle W \rangle$) be the conjunction of the following formulas:

(a) $\vDash \theta_2[u, \bar{W}^*]$,

(b) $\vDash \theta_2[u, \bar{W}]$,

(c) $(\forall X)[\varphi_{\rm nu}(u, X; \bar{W}^*) \rightarrow (\exists Y)[X \subseteq^* Y \wedge \varphi_{\rm nu}(u, Y; \bar{W})]]$.

2.12. CLAIM. Suppose $\vDash \theta_2[\bar{W}]$. $\bar{W}^+$ $\bar{D}(W)$-represents in $u$ a $Q$-name $\underset{\sim}{A}$ of a set of reals iff $\vDash \varphi_{\rm srl}(u, \bar{W}^+, \bar{W})$.

2.13. CLAIM. Suppose $\vDash \theta_2[u, \bar{W}]$, $Y$ $\bar{D}(\bar{W})$-represents the $Q$-name $\underset{\sim}{a}$ of a real in $u$, and $\bar{W}^+$ $\bar{D}(\bar{W})$-represents the $\underset{\sim}{Q}$-name $\underset{\sim}{A}$ of a set of reals in $u$.

Then for $v \subseteq u$

$$v \Vdash \text{"}\underset{\sim}{a} \in \underset{\sim}{A}\text{"} \qquad \text{iff } \varphi_{\rm mem}(u, Y, \bar{W}^+, \bar{W})$$

where $\varphi_{\rm mem}$ formalizes (d) of 2.10, i.e., 2.14. Definition. $\varphi_{\mathrm{mem}}(u, Y; \bar{W}^+, \bar{W})$ is (where $\bar{W}^+ = \bar{W}^{*\frown}\langle W\rangle$)

$$(\exists X)[\varphi_{\mathrm{ord}}(u, X; \bar{W}^*) \wedge (\forall v \subseteq u)\neg\varphi_{\mathrm{nu}}(v, X; \bar{W}^*) \wedge (\forall v \subseteq u)(\forall Z_1, Z_2)[[\varphi_{\mathrm{nu}}(v, Z_1; \bar{W}) \wedge \varphi_{\mathrm{nu}}(v, Z_2; \bar{W}^*) \wedge [Z_2 \subseteq Z_1]] \to [v \cap Z_1 \subseteq^* Y \Leftrightarrow \psi_c(v, Z_2, X; D^*, W^*_a, W)]]].$$

Proof of 2.13. Check.

2.15. Definition. We define the forcing language $L$ (for second-order theory of the continuum under the forcing $Q$) (it is a slight variant of the standard one). We have variables of three kinds: $\underset{\sim}{n}$ ($Q$-names of natural numbers), $\underset{\sim}{a}$ ($Q$-names of reals, i.e., sets of natural numbers), and $\underset{\sim}{A}$ ($Q$-names of sets of reals). We have the individual constant 0, function symbols for addition and multiplications of natural numbers, the successor relation on the natural numbers, equality between natural numbers, and two membership relations: $\underset{\sim}{n} \in \underset{\sim}{a}$, $\underset{\sim}{a} \in \underset{\sim}{A}$ (so $\underset{\sim}{a}_1 = \underset{\sim}{a}_2$ is not an atomic formula). From the atomic formulas, the formulas are generated as usual (with three kinds of quantifications).

Remark. We do not distinguish strictly between $Q$-names and variables over them. We know:

2.16. The Forcing Theorem (in this context). *For any formula* $\theta(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \ldots, \underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots) \in L$ *and* $v \in Q$, TFAE

(α) $v \Vdash_Q \theta(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \ldots, \underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots)$,

(β) *for any* $G \subseteq Q$ *generic over the universe and such that* $v \in G$, *if* $n_l = \underset{\sim}{n}_l[G]$, $a_l = \underset{\sim}{a}_l[G]$, $A_l = \underset{\sim}{A}_l[G]$ *then* $\theta[n_1, n_2, \ldots, a_1, a_2, \ldots, A_1, A_2, \ldots]$ *holds.*

2.17. Main Lemma. *For any formula* $\theta(\underset{\sim}{n}_1, \ldots, \underset{\sim}{a}_1, \ldots, \underset{\sim}{A}_1, \ldots) \in L$ *we can compute a formula* $\varphi_\theta(u, X_1, \ldots, Y_1, \ldots; \bar{W}_1^+, \ldots, \bar{W})$ *such that:*

⊕ Suppose $\Vdash_Q \theta_2[u, \bar{W}]$ *and* $X_l$ $\bar{D}(\bar{W})$*-represents the* $Q$*-name* $\underset{\sim}{n}_l$ *of a natural number in* $u$, $Y$ $\bar{D}(\bar{W})$*-represents the* $Q$*-name* $\underset{\sim}{a}_l$ *of a real in* $u$, *and* $\bar{W}_l^+$ $\bar{D}(\bar{W})$*-represents the* $Q$*-name* $\underset{\sim}{A}_l$ *of a set of reals in* $u$. Then

$$u \Vdash_Q \theta(\underset{\sim}{n}_1, \ldots, \underset{\sim}{a}_1, \ldots, \underset{\sim}{A}_1, \ldots) \quad \textit{iff} \vDash_Q \varphi_\theta[u, X_1, \ldots, Y_1, \ldots; \bar{W}_1^+, \ldots, \bar{W}].$$

Proof. By induction on $\theta$.

For atomic formulas: see 2.5 (on formulas on natural numbers), 2.9 (on $\underset{\sim}{n} \in \underset{\sim}{a}$) and 2.13 (on $\underset{\sim}{a} \in \underset{\sim}{A}$).

For Boolean combinations of atomic formulas there are no problems.

For $\theta = \forall \underset{\sim}{n}\, \theta_1$ use 2.3, 2.4 and the induction hypothesis.

For $\theta = \forall \underset{\sim}{a}\ \theta_1$ use 2.8 and the induction hypothesis.
For $\theta = \forall \underset{\sim}{A}\ \theta_1$ use 2.10A, 2.12 and the induction hypothesis.

2.18. Conclusion. For every sentence $\theta$ in the language of the second-order theory of the continuum we can compute a sentence $\varphi_\theta^*$ in the monadic theory of ${}^\omega\omega$ such that:

$$\Vdash_Q \text{“}\theta\text{”} \qquad \textit{iff} \quad M_{({}^\omega\omega)} \models \varphi_\theta^*.$$

Proof. By 2.17 there is $\varphi_\theta(\mathscr{u}, \bar{W})$ as there. Let

$$\varphi_\theta^* = (\exists \bar{W})(\forall \mathscr{u})[\theta_2(\bar{W}) \wedge \varphi_\theta(\mathscr{u}, \bar{W})].$$

As $Q$ is homogeneous and is Cohen forcing, we finish.

## §3. The combinatorics

For diversity, we do not copy [GuSh 143].

3.0. Convention. $B$ denotes a Hausdorf first countable topological space with $\leqq 2^{\aleph_0}$ open subsets (or just $\leqq 2^{\aleph_0}$ perfect subsets) (the main case is $B = {}^\omega\omega$). $A$ will denote a subset of $B$, $D = B \setminus A$. The reader can restrict himself to the case $B = {}^\omega\omega$, $A = \{\eta \in B : \eta$ not eventually constant$\}$ without great damage (just lose, e.g., non-modest subsets of ${}^\omega\omega$).

3.1. Notation. (1) $P \subseteq B$ is *perfect* in $A$ if it is closed s.t.: if $x \in \mathscr{u} \cap P$, $[x \in A \vee x$ not isolated$]$ then $(\exists P_1, P_2, \mathscr{u}_1, \mathscr{u}_2)$

$$[\mathscr{u}_1 \cap \mathscr{u}_2 = \varnothing \wedge \bigwedge_{l=1}^2 = (P_l \subseteq \mathscr{u}_l \wedge P_l \cap A \neq \varnothing \wedge P_l = \mathrm{cl}(P_l \setminus A) \wedge \mathscr{u}_l \subseteq \mathscr{u})]$$

and $P \cap A \neq \varnothing$.

(2) If $\bar{D} = \langle D_l : l < n \rangle$, we let

$$\mathrm{PR}_A^n(\bar{D}) = \{P : P \subseteq B \text{ is perfect in } A, \text{ and } \mathrm{cl}(P \cap D_l) \supseteq P \cap A \text{ for } l < n\}.$$

(3) $\mathrm{Pr}_A^n(W, \bar{D})$ with $D = \langle D_l : l < n \rangle$ as above means: there is a $P$, perfect in $A$, $A \cap P \subseteq W \cup \bigcup_l D_l$, $P \in \mathrm{PR}_A^n(\bar{D})$.

(4) In (2) and (3) we allow one to omit the superscript $n$.[†]

3.2. Convention. $\bar{D}$ denotes a finite sequence of subsets of $B \setminus A \equiv D$ such that $\mathrm{cl}(D_n) \supseteq A \cup \{x \in D : x$ not isolated$\}$.

[†] Formally, PR was not defined for an infinite sequence, but the definitions and proofs work for countable sequences; however, we do not need them as the formulas are finitary.

3.3. Definition. We say that we can separate $\{\bar{D}_i^a : i < \alpha^a\}$ from $\{\bar{D}_i^b : i < \alpha^b\}$ inside $A$, if there is $W \subseteq A$ such that

(α) for $i < \alpha^a$, $\mathrm{Pr}_A(W, \bar{D}_i^a)$,

(β) for $i < \alpha^b$, $\neg\mathrm{Pr}_A(W, \bar{D}_i^b)$.

Why does assuming CH simplify matters?

3.4. Claim (CH). Suppose $\{\bar{D}_\alpha^a : \alpha < \alpha^a\}$, $\{\bar{D}_\alpha^b : \alpha < \alpha^b\}$ are given, $|\alpha^a|, |\alpha^b| \leqq 2^{\aleph_0}$ and

$(*)_1$ if $P^a \in \mathrm{PR}_A(\bar{D}_\alpha^a)$, $P^b \in \mathrm{PR}(\bar{D}_\beta^b)$ $(\alpha < \alpha^a, \beta < \alpha^b)$ then: for $u$ an open subset of $B$, $u \cap P^b \cap A \neq \emptyset$ implies $u \cap P^b \cap A \not\subseteq P^a \cap P^b$;

$(*)_2$ $Q - D$ is not meager (as a topological space in the induced topology) when $Q \in \mathrm{PR}_A(\emptyset)$ (or even $Q \in \mathrm{PR}_A(\bar{D}_\gamma^b)$ for some $\gamma$ implies $Q \cap A$ not included in the union of $P_l \subseteq A$ $(l < \omega)$, $P_l$ perfect in $A$).

*Then* we can separate $\{D_\alpha^a : \alpha < \alpha^a\}$ from $\{D_\alpha^b : \alpha < \alpha^b\}$.

Remark. We use only the existence of a family of $\leqq 2^{\aleph_0}$ perfect separable sets which is dense enough in the family of perfect sets. This is relevant to 3.6 too.

Proof. Let $\{Q_i : i < 2^{\aleph_0}\}$ list the perfect subsets in $A$ (of $B$). We know that w.l.o.g. $\alpha^a, \alpha^b \leqq 2^{\aleph_0}$.

We choose, by induction on $\alpha$, $P_\alpha$ such that:

(a) $P_\alpha \in \mathrm{PR}_A(\bar{D}_\alpha^a)$,

(b) if $\beta, \gamma < \alpha$ and $Q_\beta \in \mathrm{PR}(\bar{D}_\gamma^b)$ then $P_\alpha \cap Q_\beta \subseteq D$.

If we succeed we shall let $W = \bigcup\{P_\alpha : \alpha < 2^{\aleph_0}\}$. Then requirement (α) of Definition 3.3 holds by demand (a). Next, (β) will hold; for, suppose $\mathrm{Pr}_A(W, \bar{D}_\gamma^b)$, so that there is $P \in \mathrm{PR}_A(\bar{D}_\gamma^b)$ such that $P \cap A \subseteq W$. But there is $\beta$ such that $P = Q_\beta$, so

$$W \cap P = \bigcup_i (P_i \cap Q_\beta) \cup D \subseteq \bigcup_{i<\beta} (P_i \cap Q_\beta) \cup D.$$

But $|\beta| \leqq \aleph_0$, and by $(*)_1$, $\bigcup_{i<\beta}(P_i \cap Q_\beta)$ is a meager subset of $Q_\beta - D$, but by the assumption above it is equal to $Q_\beta - D$, so this contradicts $(*)_2$.

The choice of $P_\alpha$ is possible, by the following claim.

3.5. Claim. If $P \in \mathrm{PR}_A^0(\emptyset)$, $D_l \subseteq P$ is not dense in $P$ for $l < l(*) < \omega$, *then* there are $\langle P_\nu : \nu \in {}^\omega 2\rangle$ such that

(a) $P_\nu \subseteq P$,

(b) $P_\nu \in \mathrm{PR}_A(\langle D_l : l < l(*)\rangle)$,

(c) $P_\nu \cap P_\eta \subseteq \bigcup_l D_l$ for $\nu \neq \eta$.

PROOF. As in [Sh 42] §7.

3.6. CLAIM. A sufficient condition for the existence of $W$ separating $L^a = \{\bar{D}^a_\alpha : \alpha < \alpha^a\}$ from $L^b = \{\bar{D}^b_\alpha : \alpha < \alpha^b\}$ inside $A$ is:

(*) there exist families $K^+$, $K^-$ of perfect subsets of $A$ such that

(i) $\mathrm{PR}_A(\bar{D}^a_\alpha) \cap K^+ \neq \emptyset$ for $\alpha < \alpha^a$;

(ii) if $Q \in \mathrm{PR}_A(\bar{D}^b_\alpha)$, $(\alpha < \alpha^b)$ *then* there is a perfect $Q' \subseteq Q$ such that $Q' \in K^-$;

(iii) if $Q \in K^-$, $P \in K^+$ then $|P \cap Q| \leqq \aleph_0$ (or even just $|P \cap Q \cap A| \leqq \kappa$, where $\kappa$ is a fixed cardinal $< 2^{\aleph_0}$);

(iv) we demand that for every $Q$ from $K^-$, $Q - D$ has cardinality $2^{\aleph_0}$.

3.6A. REMARK. If we wish, in Definition 3.1(3), replace "$A \cap P \subseteq W \cup \bigcup_l D_l$" by "$A \cap P - D \subseteq W$"; it suffices to strengthen (i) to:

(i)$_{\bar{D}}^-$ for every $\alpha < \alpha^a$ there is $P \subseteq \mathrm{PR}_A(\bar{D}^a_\alpha) \cap K^+$, $P - \bigcup_l D^a_{\alpha,l} \subseteq {}^\omega\omega - D$.

REMARK. Instead of (ii) + (iii) it is enough to require:

(ii)′ no $Q \in \mathrm{PR}_A(\bar{D}^b_\alpha)$ is included in the union of $< 2^{\aleph_0}$ many members of $K^+$.

PROOF. Let $\{Q_j : j < 2^{\aleph_0}\}$ be a list of the members of $K^-$. We choose, by induction on $\alpha < 2^{\aleph_0}$, $P_\alpha$, $P'_\alpha$ such that:

(α) $P_\alpha \in \mathrm{PR}_A(\bar{D}^a_\alpha)$, $P_\alpha \subseteq P'_\alpha \in K^+$,

(β) for $\beta < \alpha$, $Q_\beta \cap P_\alpha \subseteq D$.

In stage (α), we first choose $P'_\alpha \in K^+ \cap \mathrm{PR}_A(\bar{D}^a_\alpha)$ (use (*)(i)). Next, by Claim 3.5, there are $P_{\alpha,\eta} (\eta \in {}^\omega 2)$ such that $P_{\alpha,\eta} \subseteq P_\alpha$, $P_{\alpha,\eta} \in \mathrm{PR}_A(\bar{D}^a_\alpha)$ and $[\eta \neq \nu \Rightarrow P_{\alpha,\eta} \cap P_{\alpha,\nu} \subseteq D]$. We know that $|P'_\alpha \cap Q_\beta| \leqq \aleph_0$ for each $\beta < \alpha$ (by (*)(iii)); hence $X = \bigcup_{\beta<\alpha}(P'_\alpha \cap Q_\beta)$ has cardinality $\leqq |\alpha| + \aleph_0 < 2^{\aleph_0}$. So for some $\nu_\alpha \in {}^\omega 2$, $P_{\alpha,\nu} \cap X \subseteq D$. Now we let $P_\alpha \overset{\text{def}}{=} P_{\alpha,\nu}$.

$W = \bigcup \{P_\alpha : \alpha < 2^{\aleph_0}\}$ is as required.

3.7. CONSTRUCTION. We choose $\eta_i \in {}^\omega\omega$ for $i < 2^{\aleph_0}$ such that:

(a) for $i \neq j$, $\{k < \omega : \eta_i(k) = \eta_j(k)\}$ is a finite initial segment;

(b) $\eta_i(k) > p$ ($p$ a fixed natural number).

We then let

$$D^r_i = \{\nu \in {}^\omega\omega : \text{for every large enough } k,\ \nu(k) = \eta_i(k)\};$$

$$D^r = \bigcup_i D^r_i;$$

$$A = B \setminus D^n.$$

(c) $D^r$ contains no perfect set.

3.8. Lemma. *With $B = {}^{\omega}\omega$, let*:

$$L^a = \{\langle D_1, D_2\rangle : \text{for some } i < 2^{\aleph_0} \text{ and open } u \text{ (a subset of } A) \ D_1 \text{ and } D_2 \text{ are dense subsets of } u \cap D_i^r\};$$

$$L^b = \{\langle D_1, D_2\rangle : \text{for some open } u, D_1, D_2 \text{ are dense subsets of } D^r \cap u \text{ but for no open } u' \subseteq u \text{ and no } i < 2^{\aleph_0} \text{ are } D_1 \cap u' \subseteq D_i^r, D_2 \cap u' \subseteq D_i^r\}.$$

*Then some $W$ separates $\{\bar{D} : \bar{D} \in L^a\}$ from $\{\bar{D} : \bar{D} \in L^b\}$.*

Proof. Of course, we use the criterion (*) of 3.6.

We let for distinct $\nu_i$ †

$$k(\nu_0, \ldots, \nu_{10}) = \text{Min}\{k : \nu_0 \restriction k, \ldots, \nu_{10} \restriction k \text{ are distinct}\}.$$

Let

$$K^+ = \{P : P \in \text{PR}_A(\emptyset) \text{ and for every distinct } \nu_0, \ldots, \nu_{10} \in P \models \text{cun}(\nu_0, \ldots, \nu_{10})\}$$

where: $\models \text{cun}(\nu_0, \ldots, \nu_0)$ *iff* $\nu_0, \ldots, \nu_{10} \in A$ are distinct, and for some $i < 2^{\aleph_0}$, for every $k > k(\nu_0, \ldots, \nu_{10})$, for at most one $l \leqq 10$, $\nu_l(k) \neq \eta_i(k)$,

$$K^- = \{P : P \in \text{PR}_A(\emptyset) \text{ and for no distinct } \nu_0, \ldots, \nu_{10} \in P, \models \text{cun}(\nu_0, \ldots, \nu_{10})\}.$$

Let us check the conditions of (*) of 3.6.

*Condition* (i): So let $\bar{D} \in L^a$, $u \subseteq A$ open, $i < 2^{\aleph_0}$, $\bar{D} = \langle D_1, D_2\rangle$ and $D_1, D_2$ are dense subsets of $u \cap D_i^r$. We define by induction on $k < \omega$, $y_k, z_k, Z_k, m_k$ such that:

(1) $Z_k$ is a subset of $D_1 \cup D_2$ with exactly $k + 1$ elements,

(2) $m_k < \omega$, $m_k < m_{k+1}$,

(3) for every $\nu \in Z_k \cap (D_1 \cup D_2)$, $\nu \restriction [m_k, \omega) = \eta_i \restriction [m_k, \omega)$,

(4) for every distinct $\nu_1, \nu_2 \in Z_k$, $\nu_1 \restriction m_k \neq \nu_2 \restriction m_k$,

(5) $y_k \in Z_k$, $z_k \in Z_{k+1} - Z_k$ (so $Z_{k+1} = Z_k \cup \{z_k\}$),

(6) $z_k \restriction (m_k + 2) = y_k \restriction (m_k + 2)$, but $z_k \notin D_i^r$,

(7) for every $k$, $D' \in \{D_1, D_2\}$ and $y \in Z_k$,
for infinitely many $l > k$, $y_l = y$, $z_k \in D'$.

There are no problems in doing this, and we let

† Of course, the number 10 has no inherent significance; it just means that the author was too lazy to check the minimal number needed.

$$P \stackrel{\text{def}}{=} \mathrm{cl}\left(\bigcup_{k<\omega} Z_k\right).$$

Now $\bigcup_{k<\omega} Z_k$ is dense in itself (by (2) $m_k > k$ so by (6) and (7) this holds). Hence $P$ is perfect. Also $P - \bigcup_k Z_k$ is disjoint from $D^r$ and, as each $D_1, D_2$ is dense in $P$ (see (7)),

$$\mathrm{cl}(D_1 \cap P) = \mathrm{cl}(D_2 \cap P) = P.$$

Lastly $P \in K^+$, so $P$ is as required.

*Condition* (ii): We assume $\langle D_1, D_2\rangle \in L^b$, $Q \in \mathrm{PR}_A(\langle D_1, D_2\rangle)$. We should find a perfect $Q' \subseteq Q$, $Q' \in K^-$.

As $\langle D_1, D_2\rangle \in L^b$ there is an open $u$ such that $D_1, D_2$ are dense subsets of $u$, and for no open $u' \subseteq u$ and $i < 2^{\aleph_0}$ are $D_1 \cap u'$, $D_2 \cap u'$ dense subsets of $D^r_i$.

*Case* A: For some $i \neq j\,(<2^{\aleph_0})$ and open $u' \subseteq u$, $D_1 \cap u' \cap D^r_i$ is dense in $u' \cap Q$, $D_2 \cap u' \cap D^r_j$ is dense in $u' \cap Q$.

We define by induction on $k < \omega$ a function $h_k$ such that:

(1) $h_k : {}^k2 \to {}^{m(k)}\omega$ for some $m(k) < k$,

(2) for $\eta \in {}^k2$, $h_k(\eta) < h_k(\eta^\wedge\langle l\rangle)$ for $l = 0, 1$ (so $m(k) < m(k+1)$),

(3) $h_k(\eta^\wedge\langle 0\rangle)$, $h_k(\eta^\wedge\langle 1\rangle)$ are incomparable,

(4) $(\forall \eta \in {}^k2)(\exists \nu)[h_k(\eta) < \nu \wedge \nu \in Q \cap u']$,

(5) for every $\eta \in {}^{(k+1)}2$ there are $l_1, l_2$ such that:

(i) $\lg(h_k(\eta \upharpoonright k)) < l_1 < l_2 < \lg(h_{k+1}(\eta))$,

(ii) for no $i$, $h_{k+1}(\eta)(l_1) = \eta_i(l_1)$, $h_{k+1}(\eta)(l_2) = \eta_i(l_2)$.

There is no problem to do this. Note that if $h_k(\eta)$ is defined (and satisfies the relevant parts of (1)–(5)) then we can choose $\nu_0 \in Q$, $\eta < \nu_0$. Let $k_0$ be such that $k_0 > \lg(\eta)$ and $[k_0 \leqq k < \omega \Rightarrow \eta_i(k) \neq \eta_j(k)]$; choose $\nu_1 \in Q \cap (u' \cap D^r_i)$ such that $\nu_1 \upharpoonright k_0 = \nu_0 \upharpoonright k_0$. Let $k_1 < \omega$, $k_1 > k_0$ be such that $\nu_1(k_1) = \eta_i(k_1)$. Choose $\nu_2 \in Q \cap (u' \cap D^r_j)$, $\nu_2 \upharpoonright (k_1+1) = \nu_1 \upharpoonright (k_1+1)$, and let $k_2 < \omega$, $k_2 > k_1$, be such that $\nu_2(k_2) = \eta_j(k_2)$. Now $\nu_2 \upharpoonright (k_2+4)$ is as required from $h_{k+1}(\eta^\wedge\langle l\rangle)$ in (5).

Now $Q' = \{\nu \in {}^\omega\omega : \text{for some } \eta \in {}^\omega 2 \text{ for every } k,\ h(\eta \upharpoonright k) < \nu\}$ is as required (remembering that $\{\eta_i \upharpoonright l : 2^{\aleph_0}, l < \omega\}$ forms a tree).

*Case* B: Not case A. For some $l \in \{1, 2\}$ and open $u' \subseteq u$, for every open $u'' \subseteq u$: for infinitely many $i < 2^{\aleph_0}$, $D_l \cap u'' \cap D^r_i \neq \emptyset$.

We then define, by induction on $k < \omega$, a function $h_k$ satisfying (1)–(4) (from case A) and

(5)′ for every $k$ there is $m$ such that:

(a) for every $\eta \in {}^{k+1}2$, $\lg(h_k(\eta \upharpoonright k)) < m < \lg(h_{k+1}(\eta))$,

(b) among $\langle (h_{k+1}(\eta))[m] : \eta \in {}^{k+1}2\rangle$ there are no two which are equal.

*Condition* (iii): Let $P \in K^+$, $Q \in K^-$. We should prove that $|P \cap Q| \leqq \aleph_0$. Really checking the definitions we see that, in fact, $|P \cap Q| \leqq 11$.

*Condition* (iv): Easy.

3.9. Lemma. *For any two-place symmetric function $R$ from $2^{\aleph_0}$ to $\{ u : u \subseteq {}^{\omega}\omega$ (regular open set)$\}$, we can separate*:

$$L^a = \{\langle D_1, D_2\rangle : \text{there are } i \neq j, \text{ such that: } u \subseteq R(i,j), D_1 \text{ is a dense subset of } D_i^r \cap u \text{ and } D_2 \text{ is a dense subset of } D_j^r \cap u\},$$

$$L^b = \{\langle D_1, D_2\rangle : \text{for some open } u \text{ and } i \neq j\ (< 2^{\aleph_0}),\ u \cap R(i,j) = \varnothing \text{ and } D_1, D_2 \text{ are dense subsets of } u \cap D_i^r,\ u \cap D_j^r \text{ respectively}\},$$

*by some $W \subseteq {}^{\omega}\omega - D^r$.*

Proof. Of course, we shall use the criterion of 3.6. We let $P \in K^+$ *iff*: $P \subseteq {}^{\omega}\omega$ is perfect, and for some $i \neq j$, $P \subseteq R(i,j)$ and:

(*) for every distinct $\nu_0, \ldots, \nu_{10} \in P$:

(a) for infinitely many $k < \omega$,

$$\nu_0(k) = \nu_1(k) = \cdots = \nu_{10}(k) = \eta_i(k);$$

(b) for infinitely many $k < \omega$,

$$\nu_0(k) = \nu_1(k) = \cdots = \nu_{10}(k) = \eta_j(k);$$

(c) if $\nu_0 \restriction k, \ldots, \nu_{10} \restriction k$ are distinct then for at most one $l \leqq 10$,

$$\nu_l(k) \notin \{\eta_i(k), \eta_j(k)\}.$$

$P \in K^-$ *iff* $P \subseteq {}^{\omega}\omega$ is perfect and for some $i \neq j$,
$P \cap \mathrm{cl}(R(i,j)) = \varnothing$ and (*) above holds.
Let us check the conditions of 3.6.

*Condition* (i): The same proof as in Lemma 3.8, except that in the definition of $Z_k$ we replace condition (3) by

(3)′ (a) for every $\nu \in Z_k \cap D_1$, $\nu \restriction [m_k, \omega) = \eta_i[m_k, \omega)$,

(b) for every $\nu \in Z_k \cap D_2$, $\nu \restriction [m_k, \omega) = \eta_j \restriction [m_k, \omega)$.

*Condition* (ii): We use the proof of condition (i).

*Condition* (iii): So assume $P_1 \in K^+$, $P_2 \in K^-$. So there are $i_1 \neq j_1 < 2^{\aleph_0}$ witnessing $P_1 \in K^+$ (in particular $P_1 \subseteq R(i_1, j_1)$) and there are $i_2 \neq j_2 < 2^{\aleph_0}$ witnessing $P_2 \in K^-$ (in particular $P_2 \cap R(i_2, j_2) = \varnothing$). As $R$ is symmetric and

$\{i_1, j_1\} \neq \{i_2, j_2\}$, by symmetry assume $i_2 \notin \{i_1, j_1\}$. Suppose $|P_1 \cap P_2| \geqq 11$. Choose distinct $\nu_0, \ldots, \nu_{10} \in P_1 \cap P_2$, and we shall get a contradiction.

By the choice of $\{i_2, j_2\}$, for infinitely many $k$,

$$\nu_0(k) = \nu_1(k) = \cdots = \nu_{10}(k) = \eta_{i_2}(k).$$

But as $i_2 \notin \{i_1, j_1\}$ for every large enough $k$, $\eta_{i_2}(k) \notin \{\eta_{i_1}(k), \eta_{j_1}(k)\}$. Now by (c) of (*) of the definition of $K^+$, those two facts contradict $P \in K^+$.

*Condition* (iv): Easy.

3.10. Proof of Critical Lemma 1.2. Really, the choice of $\langle D_i^r : i < 2^{\aleph_0} \rangle$ was done. We shall write down the formulas and then 3.8 and 3.9 (via 3.7) will show that the conclusion holds (don't worry for "regular", 3.8 by 1.1A). (Use 3.8 for $\psi_a$, $\psi_b$ and 3.9 for $\psi_c$.)

$$\psi_a(u, X, D; W) \stackrel{\text{def}}{=} u \cap X \subseteq D \wedge u \subseteq \mathrm{cl}(u) \wedge (\forall X_1, X_2, v)$$

[*if* $v \subseteq u$, $X_1, X_2 \subseteq v \cap X$ are dense *then* there is a perfect $P$, $P = \mathrm{cl}(X_1 \cap v) = \mathrm{cl}(X_2 \cap v)$ and $P - (X_1 \cup X_2) \subseteq W$],

$$\psi_b(u, X, D; W) \stackrel{\text{def}}{=} \psi_a(u, X, D, W) \wedge (\forall v \subseteq u)(\forall Y)$$

[*if* $Y \subseteq v$ is dense in $v$, $Y \cap X = \emptyset$ *then* $\neg \psi_a(v, X \cup Y, D, W)$],

$$\psi_c(u, X, Y; D, W) \stackrel{\text{def}}{=} \psi_B(u, X, D, W) \wedge \psi_b(u, Y, D, W) \wedge u \cap X \cap Y = \emptyset$$

$$\wedge (\forall X_1, Y_1, v)$$

[*if* $v \subseteq u$, $X_1 \subseteq X$, $v \cap X$ is dense in $v$, $Y_1 \subseteq v \cap Y$ is dense in $v$ *then* there is a perfect $P$, $P - (X_1 \cup X_2) \subseteq W$, $P = \mathrm{cl}(P \cap X_1) = \mathrm{cl}(P \cap X_2)$].

We leave the checking to the reader.

**Concluding remarks**

What about $B \subseteq \mathbf{R}$ which is not $p$-modest? I.e. there are $D_1^*, \ldots, D_p^* \subseteq B$ such that, letting $D^* = \bigcup_{l=1}^{p} D_l^*$, $A = B \setminus D$, there are $P \in \mathrm{Pr}_A(D)$ for $B$, but for no $P \in \mathrm{Pr}_A \bar{D}$ is $P \subseteq D$. By replacing, for notational convenience, $B$ by some subspace, we get ${}^{\omega>}\omega \subseteq B \subseteq {}^{\omega\geqq}\omega$, for $l = 1, \ldots, p-1$;

$$D_l = \{\eta \in {}^{\omega>}\omega : \max(\mathrm{Rang}\, \eta) = l\},$$

$$D_p = {}^{\omega >}\omega \setminus \bigcup_{l=1}^{p-1} D_l.$$

We define $D_i^r$ $(i < r^{\aleph_0})$ as before.

There are minor changes in the proofs of 3.8 and 3.9. We replace $L^*$ by $\{D^{*\wedge}\bar{D} : \bar{D} \in L^a\}$, $K^\pm$ by $K^\pm \cap \mathrm{Pr}(\bar{D}^*)$. In the proof of condition (i) during the proof of 3.8, we add to (5):

$$\text{for } l = 1, p, \text{ for some } m \in (m_k, m_{k+1}),\ z_k \restriction m \in D_l^*.$$

We change similarly the proof of condition (ii) and of 3.9.

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